Basics of the volume-source model and its application in volcano seismology

Erhard Wielandt
Institute of Geophysics, University of Stuttgart
Richard-Wagner-Strasse 44, D – 70184 Stuttgart
e-mail: ew@geophys.uni–stuttgart.de

May 10, 2001

Contents

1 Overview and definitions

This is a collection of simple formulae describing the seismic signal from expansive or explosive spherical sources. Such sources can be characterized by the volume of displaced material, either as a function of time or as a final value. We call this the source volume. It is in general different from the total volume occupied by the source, which may depend on an arbitrarily defined interface between the source and the medium. Only changes of this volume are relevant in the present context.

A source that is defined by the time history of its volume changes will be called a volume source. It does not constitute a complete physical model for the generation of seismic signals in volcanoes since the cause of the volume changes remains unspecified. However, very simple relationships exist between the source volume and the seismic wavefield. The source volume can therefore directly be determined from seismic observations, and can serve as a constraint to any physical theory of the source itself.

Another interesting parameter of explosive sources is the total force that acts on real or virtual plane interfaces through or near the source. Simple formulae connecting the source volume, the force, and the seismic signal will also be given.

The "radial" direction is always understood as radial in space, outward from the source. In models with a free surface, the z axis points vertically downward from the source.

2 Basic solution of the elastic wave equation

The elastic wave equation is

$$\rho\ddot{\vec{u}} = \nabla P[\vec{u}] \tag{1}$$

where $\vec{u}$ is the particle displacement and P the stress tensor calculated from $\vec{u}$ and the elastic moduli. As is well known, the displacement can be decomposed into a rotation-free and a divergence-free part:

$$\vec{u} = \nabla\Phi + \nabla \times \vec{A} \tag{2}$$

where Φ is the "compressional" potential and $\vec{A}$ the "shear" potential. Φ and $\vec{A}$ are uncoupled in an inlimited homogeneous medium. Purely radial displacements such as produced by a spherical source in a homogeneous medium with the P-wave velocity c can be described by Φ alone:

$$\vec{u} = \nabla\Phi \qquad \text{with} \qquad \Phi(r,t) = -\frac{1}{r}f(t-\frac{r}{c}) \tag{3}$$

From this we get the radial displacement

$$u_r = \frac{1}{r^2}\, f(t - \frac{r}{c}) + \frac{1}{rc}\, f'(t - \frac{r}{c}) \tag{4}$$

3 Near-field and far-field

Eq. 4 decomposes the wavefield into a near-field term decaying with r^{-2} and a far-field term decaying with r^{-1}:

$$u_r = u_n + u_f, \qquad u_n = \frac{1}{r^2}\, f(t - \frac{r}{c}), \qquad u_f = \frac{1}{rc}\, f'(t - \frac{r}{c}) \tag{5}$$

u_n and u_f determine each other:

$$u_f = \frac{r}{c}\, \dot{u}_n, \qquad u_n = \frac{c}{r} \int_{-\infty}^{t} u_f(t')dt' \tag{6}$$

In case of a harmonic time dependence $u_n = u_0\, e^{j\omega t}$ we have

$$u_f = \frac{r}{c}\, j\omega u_n \tag{7}$$

and

$$\frac{|u_f|}{|u_n|} = \omega r/c = 2\pi r/\lambda \tag{8}$$

where λ is the wavelength. The far-field term thus dominates at radii greater than $1/2\pi$ wavelengths. When the deformation is quasistatic (i.e. sufficiently slow), only the near-field term is relevant.

4 The source volume and the near field

Assuming that the displacement field is generated by a spherical source of radius a, the source volume $V(t)$ is obtained from the near-field term of eq. 4 as

$$V(t) = 4\pi a^2 u_n(a,t) = 4\pi f(t - \frac{a}{c}) \tag{9}$$

Thus, the source function $f(t)$ is simply the source volume, divided by 4π. If we know the source volume as a function of time then we know the whole wavefield:

$$u_r = \frac{1}{4\pi r^2}\, V(t - \frac{r-a}{c}) + \frac{1}{4\pi rc}\, \dot{V}(t - \frac{r-a}{c}) \tag{10}$$

The remarkable thing is that we do not need to know the radius of the source, its pressure, or the elastic moduli of the medium (except for their implicit presence in the P-wave velocity c). Further, we see that the source volume is the same for each choice of a, except for the delay a/c which is in most cases negligible. The source volume can be measured at any distance from the source as long as we stay in the near field; the border between the source and the medium is arbitrary. Finally, we observe that the material behaves like an incompressible fluid in the near-field. Although our solution (4) was derived from a compressional potential, in the quasistatic limit it is not associated with a volume compression but with pure shear.

5 The source volume and the far field

The far-field term of eq. (10)

$$u_f = \frac{1}{4\pi rc}\dot{V}(t - \frac{r-a}{c}) \tag{11}$$

permits an interesting generalization. Wielandt (1975) has derived the following formula for the P-wave displacement radiated vertically downward from an explosion at the seabottom:

$$u_f(z,t) = \frac{1}{2\pi zc}\frac{Z_1}{Z_1+Z_2}\dot{V}(t - \frac{z}{c}) \tag{12}$$

Z_1 is the acoustic impedance of the water and Z_2 that of the solid; z is the depth below the seabottom. The delay time associated with the radius of the source is neglected. It can be seen that (12) is essentially the same formula as (11), except for the transmission coefficient $Z_1/(Z_1+Z_2)$ which equals $1/2$ when the two impedances are the same. The two formulae then become identical for vertical radiation. This is remarkable because eq. (12) was derived under the assumption of cylindrical rather than spherical symmetry. Also, eq. (12) remains valid even when the dynamics of the source are highly nonlinear, such as for explosions; so the same must be true for eq. (11). This simple equation can thus be used to estimate the seismic far-field of explosive sources. Of course absorption in the medium and additional transmission coefficients or spreading factors may have to be taken into account.

6 Relation with other source models

Since the primary physical cause of the seismic signal is the pressure $P(t)$ in the source, we may want to express the source volume by the pressure. The result is

$$V(t) = \frac{\pi a^3 P(t)}{\mu} \tag{13}$$

where μ is the shear modulus of the medium. Thus, we would need to know the three quantities a, P, and μ to calculate V and the seismic wavefield, and cannot determine these parameters individually from seismic observations.

Seismic sources are often idealized as point forces or couples. Both concepts are relevant to spherical explosive sources. Elementary theory gives the P-wave, far-field displacement from a vertical point force $F(t)$ in an elastic full space as

$$u_f = \frac{\cos\theta}{4\pi r(\lambda + 2\mu)}\, F(t - \frac{r}{c}) \tag{14}$$

where θ is the angle of radiation measured against the z axis. When the same force is acting on the surface of a half-space, the P-wave displacement radiated vertically downward is just twice as large.

Another source model relevant to explosions is a couple of colinear forces $\pm F(t)$ acting at a small distance d. Altough these do not have a moment in the sense of a torque, they define a generalized moment $M(t) = d\,F(t)$ and produce the far-field, P-wave displacement

$$u_f = \frac{\cos\theta}{4\pi rc(\lambda + 2\mu)}\, \dot{M}(t - \frac{r}{c}) \tag{15}$$

G. Mueller (1973) has, in analogy to the well-known definition of the seismic moment of a dislocation, defined the seismic moment of an underground nuclear explosion as

$$M_e(t) = (\lambda + 2\mu)\, S\, D(t) \tag{16}$$

where S is the area of a spherical surface around the explosion, $D(t)$ is the radial displacement at (or of) this surface, and λ and μ are the Lamé moduli. Obviously, $V(t) = SD(t)$ is the source volume, so we have

$$M_e(t) = (\lambda + 2\mu)\, V(t) \tag{17}$$

Inserting this into (15) with $\theta = 0$, we get the same result as from eq. (11). However, for spherical symmetry we must describe the explosion as the superposition of three mutually orthogonal linear dipoles, or equivalently by a moment tensor that is $M_e(t)$ times the unit tensor. The contributions from the three diagonal moments then add up to the spherical far-field term (11) in any direction.

Underwater explosions as seismic sources can, with respect to the downward radiated seismic signal, be modelled by a force $F(t)$ acting vertically on the solid halfspace. Wielandt (1975) gives the equivalent force as

$$F(t) = \frac{Z_1 Z_2}{Z_1 + Z_2}\, \dot{V}(t) \tag{18}$$

Interestingly, the main part of the vertical force does not result from the direct pressure of the explosion products on the seabottom; it rather results from the outward travelling pressure wave in the water. The force persists as long as the gas bubble expands, even when the gas pressure has already dropped below the hydrostatic equilibrium. Although the force does not act in the bubble, it can be calculated from its volume. The relationship between the bubble volume, the force, and the seismic signal is linear, even in the presence of a shock wave.

The formula may as well be applied to explosions in a homogeneous medium, by simply taking $Z_1 = Z_2 = Z$. We then have

$$F(t) = \frac{Z}{2}\dot{V}(t) \tag{19}$$

Combining this with the half-space version of eq. (14), we are back at the basic formula (11).

7 The Mogi model

The greatest deficiency of the simple concepts developed so far is that they do not allow for a free surface. Yamakawa (1955) found a static solution for the displacement field of a spherical pressure source buried in an elastic halfspace. This model for volcanic sources is usually named after Mogi (1958). Under some slight simplifications ($\lambda = \mu$, source radius $\ll$ source depth) the model predicts a purely radial displacement at the surface:

$$u_r = \frac{3a^3P}{4\mu r^2} = \frac{3V}{4\pi r^2} \tag{20}$$

It is hard to believe that the static displacement field at the free surface should differ from that in a full space (eq. 10) by nothing else than a constant factor of three. Yet the result has been checked with a finite-element computation (Kirchdoerfer 1999), and there is no doubt that it is correct. Displacements and tilts measured at the free surface can therefore be interpreted as if they were observed in a full space, except that all displacements must first be divided by a factor of three.

Using the tilt as an additional observation, the essential parameters of the source (location, depth, and volume) can theoretically be inferred from broadband seismic observations at a single station on the basis of the Mogi model (Wielandt & Forbriger 1999). The distance x_s and the depth z_s of the source are

$$x_s = \frac{3Z}{T}\left(\frac{X}{R}\right)^2, \qquad z_s = \frac{3X}{T}\left(\frac{Z}{R}\right)^2 \tag{21}$$

where X, Z, R are the displacement amplitudes in the x, z and r directions, and T is the tilt amplitude. Practically, however, the tilt should be measured over a baseline that is longer than the scale of local hererogeneities, so a small triangle of stations is more appropriate. The horizontal displacements must be separated from the tilt signal as in Wielandt & Forbriger (1999).

8 Effects of hereogeneity and topography

The effect of heterogeneity on the relatonship (10) between the source volume and the seismic displacement field is hard to predict. If there are cavities in the medium they might be expected to absorb some of the expansion of the source, but if they are more or less evenly distributed they would simply reduce the effective stiffness of the medium, and not affect the displacement vs. volume relationship. Unless there are specific reasons to assume that the material around the source is being compressed or dilated, eq. 10 seems to remain useful as a first guess for the near-field.

Heterogeneity has a marked effect on the tilt because of local strain-tilt coupling, as discussed in Wielandt & Forbriger (1999). Locally measured tilts may not represent the regional tilt at all. The latter can however be calculated from the differential vertical displacement between two broadband seismometers installed at a mutual distance in the order of 100 m.

The effect of topography on the displacements in a Mogi-type model has been investigated by Kirchdörfer (1999) with FE computations, using a conical model for the volcano. His results can be summarized as follows:

- The maximum surface displacements in the conical model are slightly larger than in the halfspace.
- The maximum vertical displacement does no longer occur vertically above the source.
- The ratio u_r/u_z is larger than in the Mogi model at horizontal distances greater than the source depth.
- At any distance, the particle motion points away from an apparent source that is below the actual one. The actual source is thus shallower than what the particle motion appears to indicate.

Looking at data from Stromboli, even the apparent source of the long-period seismic signal from explosions is found at shallow depth below the craters, so the same must be true for the real source. A vertically elongated source fits the data better than a spherical or oblate one.

References

Kirchdoerfer, M., 1999. Analysis and quasistatic FE modelling of long-period impulsive events associated with explosions at Stromboli volcano, *Annali di Geofisica*, **42**(3), 379–390.

Mogi, K., 1958. Relations between the eruptions of various volcanoes and the deformations of the ground surfaces around them, *Bull. Earthquake Res. Inst. Tokyo*, **36**, 99–134.

Mueller, G., 1973. Seismic moment and long-period radiation of underground nuclear explosions, *Bull. Seism. Soc. Am.*, **63**(3), 847–857.

Wielandt, E., 1975. Generation of seismic waves by underwater explosions, *Geophys. J. R. astron. Soc.*, **40**, 421–439.

Wielandt, E. & Forbriger, T., 1999. Near-field seismic displacement and tilt associated with the explosive activity of Stromboli, *Annali di Geofisica*, **42**(3), 407–416.

Yamakawa, N., 1955. ..., *Zisin (J. Seis. Soc. Japan)*, **8**, p. 84.