Draft

On the relationship between seismic moment and source volume

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Abstract

A nonzero isotropic moment of a seismic source does not by itself indicate a change of volume, nor does its absence indicate a constant volume. Any isotropic moment may be associated with any positive or negative volume change. A relationship between the isotropic moment and the volume change can only be established when the geometric boundary of the source is precisely defined. A definite and general relationship exists however between the isotropic moment and the volume displaced through any spherical surface enclosing the source.

1 The problem

Gerhard Müller (2001) has noted an apparent inconsistency between two relationships used to calculate the volume change ΔV of a seismic source from the isotropic part of its moment-tensor representation. The diagonal elements M of the isotropic tensor $\mathbf{M_i}$ are one-third of the trace of the original tensor $\mathbf{M} = (m_{ij})$ each.

$$\mathbf{M_i} = \begin{pmatrix} M & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & M \end{pmatrix}, \qquad M = \frac{1}{3} \sum m_{jj} \tag{1}$$

The relationships in question are

$$M = (\lambda + 2\mu/3)\ \Delta V \tag{2}$$

following Bowers and Hudson (1999) and others (for references see Müller 2001), and

$$M = (\lambda + 2\mu)\ \Delta V \tag{3}$$

following Müller (1973) and equivalent formulae for the seismic far-field derived by Wielandt (1972, 1975). Both formulae have been applied to spherical sources; however for $\mu > 0$ eq. (2) gives a larger source volume than eq. (3). The apparent inconsistency has not fully been resolved in Müller's note; he shows however that (2) applies to the moment-tensor equivalent of a crack and (3) to the moment-tensor equivalent of a spherical explosive source.

2 Transformational and distributed volume changes

A hint to the origin of the discrepancy is found in Aki and Richard's textbook (1980), section 3.4, where eq. (2) is derived with the specification that ΔV is the "transformational", i.e. hypothetical isobaric, volume change. Actually the material in the source does not experience isobaric conditions because it must displace the surrounding medium, which counteracts with additional forces and moments. The transformational and the actual change of volume are in general different. This has apparently escaped some readers of the textbook. The two volumes coincide only for cracks, as we shall see in a moment.

Another complication is that the displacement field $\vec{u}$ of a seismic source is in general not divergence-free ($\nabla\vec{u} \neq 0$), so the volume change depends on where we measure it. As the interface between the "source" and the "medium" is to some extent arbitrary, so is the volume change.

Being cautioned against these two pitfalls, we can now proceed to a quantitative discussion of some simple sources. **We consider only the permanent displacement field associated with the final value of the moment tensor, and only homogeneous and isotropic media.**

3 Definition of source and moment

Our arguments in the following sections depend on how the seismic source and its moment are defined. We give conceptual definitions, without attempting to be mathematically strict.

Our source is a finite volume in which intrinsic processes occur so that the homogeneous elastic equation of motion $\rho\ddot{\vec{u}} = \nabla\sigma$ is violated; σ is the stress tensor calculated from $\vec{u}$ with a linear and time-invariant law. The violation may be restricted to part of the source volume, to a crack or several cracks inside this volume, etc. The action of the source on the surrounding medium can be replaced by stresses acting on the boundary of a cavity that has the same shape as the source. If either these stresses or the associated displacements at the boundary are known, the seismic deformation of the outer medium is uniquely determined, and so are the displacements resp. stresses that were not known a priori.

The seismic moment tensor is not the moment tensor of forces acting at the boundary of a cavity; it is the moment tensor of those forces that must be applied to the same mathematical surface in an elastic full medium to produce the observed deformation outside the source. (This is, of course, just one of many equivalent definitions of the moment tensor.) I take it for granted by general theory that a unique relationship exists between the moment tensor and the far-field deformation of the outer medium. In order to model the near-field correctly, one would have to include higher-order moments as coefficients in a suitable series expansion of the wavefield.

Using the concept of forces acting on a mathematical boundary in a full space, we can split the forces and their moments into one contribution that deforms the outer medium and one that deforms the interior of the source (not necessarily in the same way as in the real source). For this purpose we consider the boundary as a double surface, one surface forming the inner boundary of the outer medium, the other vice versa. We let act forces on the outer medium so that it is properly deformed, and on the inner medium so that the gap between the two surfaces is closed. Both sets of forces are uniquely determined by the deformation of the boundary, and the same holds for their moments.

4 Isolated bodies

Any finite elastic body subject to a homogeneous isotropic stress (a negative pressure) σ will change its volume by

$$\Delta V = \sigma V/(\lambda + 2\mu/3) \tag{4}$$

where $(\lambda + 2\mu/3)$ is the bulk modulus. In this case, the external forces have an isotropic moment $M = \sigma V$, and eq. (4) is formally the same as eq. (2). Since uniform shear does not change the volume of an isotropic body, the volume change of any finite body under homogeneous stress is described by eq. (4).

5 Crack-equivalent sources

The correspondence between equations (4) and (2) permits an interesting observation on idealized cracks. The relationship between moment and volume is the same for an isolated crack-shaped body and for the moment-tensor equivalent of a crack (this is, a suitable distribution of moments in an unbroken full space). Thus, all of the moment is needed to dilate the crack filling, and no moment remains to push the surrounding medium away. Opening a crack in a full space requires no moment. This may sound paradox, but is not. The forces (stresses) opening the crack are finite for a finite crack but have a negligible moment when the crack is thin.

6 Spherical sources

An elementary calculation shows that the actual volume change of an expanding spherical source desribed by the moment tensor (1) is given by eq. (3) while eq. (2) defines the transformational volume change, which is larger. It is instructive to look at the moments required for a given expansion ΔV. The total moment $M = (\lambda + 2\mu)\Delta V$ results from a moment $M = (\lambda + 2\mu/3)\Delta V$ that expands the interior of the sphere, and a moment $M = (4\mu/3)\Delta V$ that pushes the surrounding medium away. The outer displacement field is divergence-free both in liquids and in solids, so the volume change of the spherical source can be measured at any distance and over any closed surface.

7 Prolate sources

For a LVD (linear vector dipole) type source that is prolate in the direction of the forces, most of the moment is required to deform the outer medium, and only a small fraction remains to expand the source itself. The medium counteracts the lengthwise expansion of a thin and long body very strongly. The ratio between the moment and the volume change of the source can be arbitrarily large, as is apparent from the following model: Two colinear forces of opposite direction act at some finite distance d from each other. To avoid singularities of the displacement field, we distribute each force over a small sphere with a radius $r \ll d$. The source volume is defined as comprising the two spheres and a narrow tube connecting them. Obviously, the volume change within the source can be made arbitrarily small. Most of the volume change occurs outside the source. Thus, the ratio of moment to source-volume change can be arbitrarily large.

8 The volume displacement through spherical surfaces

A source with the isotropic moment M displaces the volume

$$\Delta V = M/(\lambda + 2\mu) \tag{5}$$

through any spherical surface that encloses the source. This result is obtained in four steps.

1 - The displacement field of a linear vector dipole (LVD, a point source representing two equal colinear forces of opposite sign) can be calculated analytically. The radial component of the displacement can then be integrated over the surface of a sphere centered at the source. The result is given by eq. (5), independent of the radius of the sphere. The derivation is based on formulae for the displacement field of a point force taken from Gerhard Müller's lecture notes on

the theory of elastic waves (1986), and was doublechecked with the computer-algebra system MuPAD Pro 2.5 (2002).

2 - The result can be generalized to spherical surfaces that are not centered around the point source but enclose it. The result is the same as for centered spheres. A mathematical proof is given in the appendix. The result was also confirmed by numerical integration with random positions of the source within the sphere.

3 - Eq. (5) being true for any position of a point source within a given sphere, it remains valid for distributed sources.

4 - An arbitrary moment tensor is the combination of three LVDs along its principal axes, so (5) is valid for any moment-tensor source.

Equation (5) appears to be the most general moment-to-volume relationship that can be derived without a detailed knowledge of the source geometry. It retains its physical meaning for point sources, for which a "volume change" is otherwise difficult to define.

9 Source volume and moment

We have seen that he volume change of an opening crack is larger than the associated volume displacement through a spherical surface (section 5). For an expanding spherical source the two volumes are identical, so we do not have to distinguish between different definitions of the source volume. A source that is prolate in the direction of colinear forces has a smaller volume change than it displaces through a spherical surface. The difference is made up by the compression or expansion of the medium around the source, due to the moments acting in the source. By combining prolate and oblate sources of opposite polarity, sources may be constructed where the volume change of the source itself and the volume displacement through a spherical surface are of opposite sign.

Together with eq. (5), this implies that the isotropic part of the moment tensor may not even indicate the sign of the volume change of the source. We demonstrate this with a simple example. Consider a situation where incompressible magma is stored in a spherical chamber and then injected into a dike. The total volume of the cavities remains constant; however the isotropic moment of the chamber decreases by $M = (\lambda + 2\mu)\Delta V$ while the isotropic moment of the dike, idealized as a crack, increases by $M = (\lambda + 2\mu/3)\Delta V$. An isotropic moment $M = (4\mu/3)\Delta V$ disappears in the process; this quantity would be inferred from seismic observations as the isotropic moment of the composite source. A conventional interpretation would suggest a reduction of the magma volume, yet actually the volume reduction has occurred in the surrounding medium. The walls of the dike - which are not part of the source - were compressed while the walls of the chamber retained their volume. Alternatively, we might consider the case that the magma is compressible, and expands in the process so that the compression of the dike walls is made up. Despite this expansion of the source, the overall volume is constant and the isotropic moment is zero. Intermediate between the two cases we find expanding sources whose isotropic moment is negative. This example does not appear to be far-fetched, and should caution against an uncritical application of any moment-to-volume relationship.

10 A mathematical source model

For a given isotropic moment M, we can easily construct a mathematical source model with any desired positive or negative volume change. We simply let a linear vector dipole (LVD) with the moment $3M$ act in a small sphere, either in its center or evenly distributed. (If the isotropic moment equals zero, we may use a suitable scaled CLVD - a compensated linear vector dipole - instead.) The displacement field outside the source sphere has the divergence

$$\nabla\vec{u} = 3M\frac{1 - 3\cos^2(\theta)}{4\pi(\lambda + 2\mu)\, r^3} \tag{6}$$

The volume integral of this divergence is zero over any spherical shell enclosing the source, but since the divergence decays as r^{-3} in any given direction from the source, the integral diverges to plus or minus infinity if taken over a suitable cone in the polar or equatorial direction. Our definition of the source permits to consider any part of such a cone as part of the source. We can thus define the boundary of the source in such a way that its total volume change has any desired value.

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13 Appendix

We want to prove the following theorem:

If $\vec{u}(r,\theta,\phi)$ is a displacement field whose divergence $q(r,\theta,\phi) = \nabla\vec{u}(r,\theta,\phi)$ has the following properties:

- *q fulfills the Laplace equation $\Delta q = 0$ outside a finite source region*
- *q tends uniformly to zero for $r \to \infty$ after multiplication with r^2*

then $\vec{u}$ displaces the same volume through any spherical surface enclosing the source region.

We note that the static displacement field of a moment-tensor source in a homogeneous isotropic full space has the required properties, as can be analytically derived from the explicit solution (the use of computer algebra is recommended). For a LVD source q is given by eq. (6). It is a multiple of the classical dipole potential, $P_2(\cos\theta)/r^3$, and therefore a solution of $\Delta q = 0$. For our present purpose we need the theorem only for point sources, but may as well prove it for distributed sources at the same time.

The proof of the theorem is as follows:

Under the above conditions, q can be expanded into a series of spherical harmonics associated with negative powers of the radius:

$$q(r,\theta,\phi) = \sum_{\ell=2}^{\infty} \sum_{m=-\ell}^{\ell} c_{\ell m}\, r^{-\ell-1}\, P_\ell^m(\cos\theta)\, e^{im\phi} \tag{7}$$

The origin may be chosen arbitrarily. The expansion will converge outside the smallest sphere around the origin that encloses the source. **The condition that spherical surfaces enclose the source region will from now on be understood.** The $\ell = 0$ and $\ell = 1$ terms are absent for any choice of the origin due to our assumption on the asymptotic behaviour of q for large radii. The integral or average of q over any spherical surface, which is related to the $\ell = 0$ term of the expansion, is therefore zero. Generally, any solution of the Laplace equation that decays faster than r^{-1} for $r \to \infty$ must have zero average over any spherical surface enclosing the source. The same result can be derived without invoking the series expansion by setting up Green's formula

$$\int_{Volume} (u\Delta v - v\Delta u)\, dV = \int_{Surface} (u\nabla v - v\nabla u)\, d\vec{S} \qquad (8)$$

with $u = q$ and $v = 1/r$, and integrating over a spherical shell whose outer radius is then made infinite.

Consequently, if we integrate q over a spherical shell of any thickness, the result must be zero.

Since q is the divergence of $\vec{u}$, this implies that $\vec{u}$ displaces the same total volume through the inner and outer surfaces of the shell. The volume displacement through all spherical surfaces having a common center is thus the same.

We now consider two such sets of surfaces. We select surface S_1 from the first set and S_2 from the second set so that they have the same radius r. The surfaces intersect because they both enclose the source. Let the volumes displaced by $\vec{u}$ through these surfaces be D_1 and D_2. The difference $D_2 - D_1$ may be calculated by integrating q over the differential volume enclosed between S_1 and S_2 (one of the two bowl-shaped halves of this volume enters with a negative sign). By making r sufficiently large, $D_2 - D_1$ can be made arbitrarily small because the volume of integration increases with r^2 while the integrand tends uniformly to zero faster than r^{-2}. On the other hand, the displaced volume is constant within each set, so D_1 must equal D_2 for any value of r. The displaced volume is thus the same for all surfaces of both sets. This completes our proof.